

ON MEAN ERGODIC CONVERGENCE IN THE CALKIN ALGEBRAS

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ABSTRACT. In this paper, we give a geometric characterization of mean ergodic convergence in the Calkin algebras for Banach spaces that have the bounded compact approximation property.

1. INTRODUCTION

Let X be a real or complex Banach space and let $B(X)$ be the algebra of all bounded linear operators on X . Suppose that $T \in B(X)$ and consider the sequence

$$M_n(T) := \frac{I + T + \dots + T^n}{n+1}, \quad n \geq 1.$$

In [3], Dunford considered the norm convergence of $(M_n(T))_n$ and established the following characterizations.

Theorem 1.1. *Suppose that X is a complex Banach space and that $T \in B(X)$ satisfies $\frac{\|T^n\|}{n} \rightarrow 0$. Then the following conditions are equivalent.*

- (1) $(M_n(T))_n$ converges in norm to an element in $B(X)$.
- (2) 1 is a simple pole of the resolvent of T or 1 is in the resolvent set of T .
- (3) $(I - T)^2$ has closed range.

It was then discovered by Lin in [6] that $I - T$ having closed range is also an equivalent condition. Moreover, Lin's argument worked also for real Banach spaces. This result was later improved by Mbekhta and Zemánek in [9] in which they showed that $(I - T)^m$ having closed range, where $m \geq 1$, are also equivalent conditions. More precisely,

Theorem 1.2. *Let $m \geq 1$. Suppose that X is a real or complex Banach space and that $T \in B(X)$ satisfies $\frac{\|T^n\|}{n} \rightarrow 0$. Then the sequence $(M_n(T))_n$ converges in norm to an element in $B(X)$ if and only if $(I - T)^m$ has closed range.*

Let $K(X)$ be the closed ideal of compact operators in $B(X)$. If $T \in B(X)$ then its image in the Calkin algebra $B(X)/K(X)$ is denoted by \hat{T} . By Dunford's Theorem

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1.1 or by an analogous version for Banach algebras (without condition (3)), when X is a complex Banach space and $\frac{\|\dot{T}^n\|}{n} \rightarrow 0$, the convergence of $(M_n(\dot{T}))_n$ in the Calkin algebra is equivalent to 1 being a simple pole of the resolvent of \dot{T} or being in the resolvent set of \dot{T} . But even if we are given that the limit $\dot{P} \in B(X)/K(X)$ exists, there is no obvious geometric interpretation of \dot{P} . In the context of Theorems 1.1 and 1.2, if the limit of $(M_n(T))_n$ exists, then it is a projection onto $\ker(I - T)$. In the context of the Calkin algebra, the limit \dot{P} is still an idempotent in $B(X)/K(X)$; hence by making a compact perturbation, we can assume that P is an idempotent in $B(X)$ (see Lemma 2.6 below).

A natural question to ask is: what is the range of P ? Although the range of P is not unique (since P is only unique up to a compact perturbation), it can be thought of as an analog of $\ker(I - T)$ in the Calkin algebra setting. If $T_0 \in B(X)$ then $\ker T_0$ is the maximal subspace of X on which $T_0 = 0$. This suggests that the analog of $\ker T_0$ in the Calkin algebra setting is the maximal subspace of X on which T_0 is compact. But the maximal subspace does not exist unless it is the whole space X . Thus, we introduce the following concept.

Let X be a Banach space and let (P) be a property that a subspace M of X may or may not have. We say that a subspace $M \subset X$ is an *essentially maximal* subspace of X satisfying (P) if it has (P) and if every subspace $M_0 \supset M$ having property (P) satisfies $\dim M_0/M < \infty$.

Then the analog of $\ker T_0$ in the Calkin algebra setting is an essentially maximal subspace of X on which T_0 is compact. It turns that if such an analog for $I - T$ exists, then it is already sufficient for the convergence of $(M_n(\dot{T}))_n$ in the Calkin algebra (at least for a large class of Banach spaces), which is the main result of this paper.

Before stating this theorem, we recall that a Banach space Z has the *bounded compact approximation property* (BCAP) if there is a uniformly bounded net $(S_\alpha)_{\alpha \in \Lambda}$ in $K(Z)$ converging strongly to the identity operator $I \in B(Z)$. It is always possible to choose Λ to be the set of all finite dimensional subspaces of Z directed by inclusion. If the net $(S_\alpha)_{\alpha \in \Lambda}$ can be chosen so that $\sup_{\alpha \in \Lambda} \|S_\alpha\| \leq \lambda$, then we say that Z has the λ -BCAP. It is known that if a reflexive space has the BCAP, then the space has the 1-BCAP. For $T \in B(X)$, the essential norm $\|T\|_e$ is the norm of \dot{T} in $B(X)/K(X)$.

Theorem 1.3. *Let $m \geq 1$. Suppose that X is a real or complex Banach space having the bounded compact approximation property. If $T \in B(X)$ satisfies $\frac{\|T^n\|_e}{n} \rightarrow 0$, then the following conditions are equivalent.*

- (1) *The sequence $(M_n(\dot{T}))_n$ converges in norm to an element in $B(X)/K(X)$.*
- (2) *There is an essentially maximal subspace of X on which $(I - T)^m$ is compact.*

The idea of the proof is to reduce Theorem 1.3 to Theorem 1.2 by constructing a Banach space \hat{X} and an embedding $f : B(X)/K(X) \rightarrow B(\hat{X})$ so that if $T \in B(X)$ and if there is an essentially maximal subspace M of X on which T is compact, then $f(\dot{T})$ has closed range, and then applying Theorem 1.2 to $f(\dot{T})$. The BCAP of X is used to show that f is an embedding but is not used in the construction of \hat{X} and f . The construction of f is based on the Calkin representation [1, Theorem 5.5].

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2. THE CALKIN REPRESENTATION FOR BANACH SPACES

In this section, X is a fixed infinite dimensional Banach space. Let Λ_0 be the set of all finite dimensional subspaces of X directed by inclusion \subset . Then $\{\{\alpha \in \Lambda_0 : \alpha \supset \alpha_0\} : \alpha_0 \in \Lambda_0\}$ is a filter base on Λ_0 , so it is contained in an ultrafilter U on Λ_0 .

Let Y be an arbitrary infinite dimensional Banach space and let $(Y^*)^U$ be the ultrapower (see e.g., [2, Chapter 8]) of Y^* with respect to U . (The ultrafilter U and the directed set Λ_0 do not depend on Y .) If $(y_\alpha^*)_{\alpha \in \Lambda_0}$ is a bounded net in Y^* , then its image in $(Y^*)^U$ is denoted by $(y_\alpha^*)_{\alpha, U}$. Consider the (complemented) subspace

$$\widehat{Y} := \left\{ (y_\alpha^*)_{\alpha, U} \in (Y^*)^U : w^*\text{-}\lim_{\alpha, U} y_\alpha^* = 0 \right\}$$

of $(Y^*)^U$. Here $w^*\text{-}\lim_{\alpha, U} y_\alpha^*$ is the w^* -limit of $(y_\alpha^*)_{\alpha \in \Lambda_0}$ through U , which exists by the Banach-Alaoglu Theorem.

Whenever $T \in B(X, Y)$, we can define an operator $\widehat{T} \in B(\widehat{Y}, \widehat{X})$ by sending $(y_\alpha^*)_{\alpha, U}$ to $(T^*y_\alpha^*)_{\alpha, U}$. Note that if $K \in K(X, Y)$ then $\widehat{K} = 0$, where $K(X, Y)$ denotes the space of all compact operators in $B(X, Y)$.

Theorem 2.1. *Suppose that X has the λ -BCAP. Then the operator $f : B(X)/K(X) \rightarrow B(\widehat{X})$, $\dot{T} \mapsto \widehat{T}$, is a norm one $(\lambda + 1)$ -embedding into $B(\widehat{X})$ satisfying*

$$f(\dot{I}) = I \text{ and } f(\dot{T}_1 \dot{T}_2) = f(\dot{T}_2) f(\dot{T}_1), \quad T_1, T_2 \in B(X).$$

Proof. It is easy to verify that f is a linear map, $f(\dot{I}) = I$, and $f(\dot{T}_1 \dot{T}_2) = f(\dot{T}_2) f(\dot{T}_1)$ for $T_1, T_2 \in B(X)$. If $T \in B(X)$, then clearly $\|f(\dot{T})\| \leq \|T\|$, and thus we also have $\|f(\dot{T})\| \leq \|T\|_e$. Hence $\|f\| \leq 1$. It remains to show that f is a $(\lambda + 1)$ -embedding (i.e., $\inf_{\|T\|_e > 1} \|f(\dot{T})\| \geq (\lambda + 1)^{-1}$).

To do this, let $T \in B(X)$ satisfy $\|T\|_e > 1$. Since X has the λ -BCAP, we can find a net of operators $(S_\alpha)_{\alpha \in \Lambda_0} \subset K(X)$ converging strongly to I such that $\sup_{\alpha \in \Lambda_0} \|S_\alpha\| \leq \lambda$. Then $\|T^*(I - S_\alpha)^*\| = \|(I - S_\alpha)T\| \geq \|T\|_e > 1$, $\alpha \in \Lambda_0$. Thus, there exists $(x_\alpha^*)_{\alpha \in \Lambda_0} \subset X^*$ such that $\|x_\alpha^*\| = 1$ and $\|T^*(I - S_\alpha)^*x_\alpha^*\| > 1$ for $\alpha \in \Lambda_0$.

Note that for every $x \in X$,

$$\limsup_{\alpha \in \Lambda_0} |\langle (I - S_\alpha)^*x_\alpha^*, x \rangle| = \limsup_{\alpha \in \Lambda_0} |\langle x_\alpha^*, (I - S_\alpha)x \rangle| \leq \limsup_{\alpha \in \Lambda_0} \|(I - S_\alpha)x\| = 0,$$

and so the net $((I - S_\alpha)^*x_\alpha^*)_{\alpha \in \Lambda_0}$ converges in the w^* -topology to 0. By the construction of U , this implies that

$$w^*\text{-}\lim_{\alpha, U} (I - S_\alpha)^*x_\alpha^* = 0.$$

Therefore, due to the definition $f(\dot{T}) = \widehat{T}$, we obtain

$$\begin{aligned} (1 + \lambda)\|f(\dot{T})\| &\geq \|f(\dot{T})\| \lim_{\alpha, U} \|(I - S_\alpha)^*x_\alpha^*\| &= \|f(\dot{T})\| \|((I - S_\alpha)^*x_\alpha^*)_{\alpha, U}\| \\ &\geq \|f(\dot{T})\| \|((I - S_\alpha)^*x_\alpha^*)_{\alpha, U}\| \\ &= \lim_{\alpha, U} \|T^*(I - S_\alpha)^*x_\alpha^*\| \geq 1. \end{aligned}$$

It follows that $\|f(\dot{T})\| \geq (1 + \lambda)^{-1}$ whenever $\|T\|_e > 1$. \square

Remark 1. We do not know whether Theorem 2.1 is true without the hypothesis that X has the BCAP.

Remark 2. The embedding in Theorem 2.1 is an isometry if the approximating net can be chosen so that $\|I - S_\alpha\| = 1$ for every α . This is the case if, for example, the space X has a 1-unconditional basis. However, we do not know whether the embedding is an isometry if $X = L_p(0, 1)$ with $p \neq 2$.

If N is a subset of Y^* , then we can define a subset N' of \widehat{Y} by

$$N' := \left\{ (y_\alpha^*)_{\alpha, U} \in \widehat{Y} : \lim_{\alpha, U} d(y_\alpha^*, N) = 0 \right\},$$

where

$$d(y_\alpha^*, N) := \inf_{z^* \in N} \|y_\alpha^* - z^*\|.$$

Lemma 2.2. *If N is a w^* -closed subspace of Y^* , then for every $(y_\alpha^*)_{\alpha, U} \in \widehat{Y}$,*

$$d((y_\alpha^*)_{\alpha, U}, N') \leq 2 \lim_{\alpha, U} d(y_\alpha^*, N).$$

Proof. Let $a = \lim_{\alpha, U} d(y_\alpha^*, N)$. Let $\delta > 0$. Then

$$A := \{\alpha \in \Lambda : d(y_\alpha^*, N) < a + \delta\} \in U.$$

Whenever $\alpha \in A$, $\|y_\alpha^* - z_\alpha^*\| < a + \delta$ for some $z_\alpha^* \in N$. If we take $z_\alpha^* = 0$ for $\alpha \notin A$, then, since $\sup_{\alpha \in \Lambda} \|y_\alpha^*\| < \infty$,

$$\sup_{\alpha \in \Lambda} \|z_\alpha^*\| = \sup_{\alpha \in A} \|z_\alpha^*\| \leq (a + \delta) + \sup_{\alpha \in A} \|y_\alpha^*\| < \infty.$$

As a consequence, $\left(z_\alpha^* - w^* \lim_{\beta, U} z_\beta^* \right)_{\alpha, U} \in N'$, since N is w^* -closed. Therefore,

$$\begin{aligned} d((y_\alpha^*)_{\alpha, U}, N') &\leq d\left((y_\alpha^*)_{\alpha, U}, \left(z_\alpha^* - w^* \lim_{\beta, U} z_\beta^*\right)_{\alpha, U}\right) \\ &= \lim_{\alpha, U} \left\| y_\alpha^* - z_\alpha^* + w^* \lim_{\beta, U} z_\beta^* \right\| \\ &\leq \lim_{\alpha, U} \|y_\alpha^* - z_\alpha^*\| + \left\| w^* \lim_{\beta, U} z_\beta^* \right\| \\ &\leq (a + \delta) + \left\| w^* \lim_{\beta, U} (z_\beta^* - y_\beta^*) \right\| \\ &\leq (a + \delta) + \lim_{\beta, U} \|z_\beta^* - y_\beta^*\| \leq 2(a + \delta). \end{aligned}$$

But δ can be arbitrarily close to 0 so $d((y_\alpha^*)_{\alpha, U}, N') \leq 2a = 2 \lim_{\alpha, U} d(y_\alpha^*, N)$. \square

Proposition 2.3. *If X and Y are infinite dimensional Banach spaces and if $T \in B(X, Y)$ has closed range then $\widehat{T} \in B(\widehat{Y}, \widehat{X})$ also has closed range.*

Proof. The operator T has closed range so T^* also has closed range. Let $c = \inf\{\|T^*y^*\| : y^* \in Y^*, d(y^*, \ker T^*) = 1\} > 0$. Then by Lemma 2.2, for every $(y_\alpha^*)_{\alpha,U} \in \widehat{Y}$,

$$\|\widehat{T}(y_\alpha^*)_{\alpha,U}\| = \lim_{\alpha,U} \|T^*y_\alpha^*\| \geq c \lim_{\alpha,U} d(y_\alpha^*, \ker T^*) \geq \frac{c}{2} d((y_\alpha^*)_{\alpha,U}, (\ker T^*)').$$

But obviously $(\ker T^*)' \subset \ker \widehat{T}$, and so

$$\|\widehat{T}(y_\alpha^*)_{\alpha,U}\| \geq \frac{c}{2} d((y_\alpha^*)_{\alpha,U}, \ker \widehat{T}), \quad (y_\alpha^*)_{\alpha,U} \in \widehat{Y}.$$

Hence \widehat{T} has closed range. \square

Lemma 2.4. *Suppose that $X \subset Y$ and that $T \in B(X)$. Let $T_0 \in B(X, Y)$, $x \mapsto Tx$. Then $\widehat{T_0}\widehat{Y} = \widehat{T}\widehat{X}$.*

Proof. If $(y_\alpha^*)_{\alpha,U} \in \widehat{Y}$, then for each $\alpha \in \Lambda$, we have $T_0^*y_\alpha^* = T^*(y_{\alpha|X}^*)$, and $(y_{\alpha|X}^*)_{\alpha,U} \in \widehat{X}$. Thus $\widehat{T_0}(y_\alpha^*)_{\alpha,U} = (T_0^*y_\alpha^*)_{\alpha,U} = (T^*(y_{\alpha|X}^*))_{\alpha,U} = \widehat{T}(y_{\alpha|X}^*)_{\alpha,U} \in \widehat{T}\widehat{X}$. Hence $\widehat{T_0}\widehat{Y} \subset \widehat{T}\widehat{X}$.

Conversely, if $(x_\alpha^*)_{\alpha,U} \in \widehat{X}$ then we can extend each x_α^* to an element $y_\alpha^* \in Y^*$ such that $\|y_\alpha^*\| = \|x_\alpha^*\|$. Thus we have $\left(y_\alpha^* - w^*\text{-}\lim_{\beta,U} y_\beta^*\right)_{\alpha,U} \in \widehat{Y}$. Note that

$$T_0^* \left(w^*\text{-}\lim_{\beta,U} y_\beta^* \right) = w^*\text{-}\lim_{\beta,U} T_0^* y_\beta^* = w^*\text{-}\lim_{\beta,U} T^* x_\beta^* = T^* \left(w^*\text{-}\lim_{\beta,U} x_\beta^* \right) = 0.$$

This implies that

$$\begin{aligned} \widehat{T}(x_\alpha^*)_{\alpha,U} = (T^*x_\alpha^*)_{\alpha,U} &= (T_0^*y_\alpha^*)_{\alpha,U} \\ &= \left(T_0^* \left(y_\alpha^* - w^*\text{-}\lim_{\beta,U} y_\beta^* \right) \right)_{\alpha,U} \\ &= \widehat{T_0} \left(y_\alpha^* - w^*\text{-}\lim_{\beta,U} y_\beta^* \right)_{\alpha,U} \in \widehat{T_0}\widehat{Y}. \end{aligned}$$

Therefore $\widehat{T}\widehat{X} \subset \widehat{T_0}\widehat{Y}$. \square

Proposition 2.5. *Suppose that $T \in B(X)$ and that there exists an essentially maximal subspace M of X on which T is compact. Then \widehat{T} has closed range.*

Proof. Without loss of generality, we may assume that X is a subspace of $Y = \ell_\infty(J)$ for some set J . Define $T_0 \in B(X, \ell_\infty(J))$, $x \mapsto Tx$. Then by assumption, there is an essentially maximal subspace M of X on which T_0 is compact. By [7, Theorem 3.3], there exists $K \in K(X, \ell_\infty(J))$ such that $K|_M = T_0|_M$.

We now show that $T_0 - K \in B(X, \ell_\infty(J))$ has closed range. Since $M \subset \ker(T_0 - K)$ and M is an essentially maximal subspace of X on which $T_0 - K$ is compact, $\ker(T_0 - K)$ is an essentially maximal subspace of X on which $T_0 - K$ is compact.

Let π be the quotient map from X onto $X/\ker(T_0 - K)$. Define the (one-to-one) operator $R : X/\ker(T_0 - K) \rightarrow \ell_\infty(J)$, $\pi x \mapsto (T_0 - K)x$. If R does not have closed range, then by [8, Proposition 2.c.4], R is compact on an infinite dimensional subspace V of $X/\ker(T_0 - K)$. Hence, $T_0 - K$ is compact on $\pi^{-1}V$ and so by the essential maximality of $\ker(T_0 - K)$, we have $\dim \pi^{-1}V/\ker(T_0 - K) < \infty$. Thus, $V = \pi^{-1}V/\ker(T_0 - K)$ is finite dimensional, which contradicts the definition of V .

Therefore, R has closed range and so $T_0 - K$ also has closed range. By Proposition 2.3, $\widehat{T_0 - K}$ has closed range. But $\widehat{K} = 0$ so $\widehat{T_0}$ has closed range and by Lemma 2.4, \widehat{T} has closed range. \square

Lemma 2.6. *Suppose that $P \in B(X)$ and that \dot{P} is an idempotent in $B(X)/K(X)$. Then P is the sum of an idempotent in $B(X)$ and a compact operator on X .*

Proof. We first treat the case where the scalar field is \mathbb{C} . From Fredholm theory (see e.g. [5, Chapters XI and XVII]), we know that since $\sigma(\dot{P}) \subset \{0, 1\}$, the only possible cluster points of $\sigma(P)$ are 0 and 1. Thus, there exists $0 < r < 1$ such that $\{z \in \mathbb{C} : |z - 1| = r\} \cap \sigma(P) = \emptyset$. Then $\dot{P} = \frac{1}{2\pi i} \oint_{|z-1|=r} (zI - \dot{P})^{-1} dz$ and so $P - \frac{1}{2\pi i} \oint_{|z-1|=r} (zI - P)^{-1} dz \in K(X)$. But $\frac{1}{2\pi i} \oint_{|z-1|=r} (zI - P)^{-1} dz$ is an idempotent in $B(X)$ (see e.g. [10, Theorem 2.7]). This completes the proof in the complex case.

If X is a real Banach space, then let X_C and P_C be the complexifications (see [4, page 266]) of X and P , respectively. Thus, \dot{P}_C is an idempotent in $B(X_C)/K(X_C)$. Since the only possible cluster points of $\sigma(P_C)$ are 0 and 1, there exists a closed rectangle R in the complex plane symmetric with respect to the real axis such that 1 is in the interior of R , 0 is in the exterior of R , and $\sigma(P_C)$ is disjoint from the boundary ∂R of R . By [4, Lemma 3.4], the idempotent $\frac{1}{2\pi i} \oint_{\partial R} (zI - P_C)^{-1} dz$ in $B(X_C)$ is induced by an idempotent P_0 in $B(X)$. Since $P_C - \frac{1}{2\pi i} \oint_{\partial R} (zI - P_C)^{-1} dz \in K(X_C)$, we see that $P - P_0 \in K(X)$. \square

Proof of Theorem 1.3. “(1) \Rightarrow (2)”: Let $\dot{P} := \lim_{n \rightarrow \infty} \frac{\dot{I} + \dot{T} + \dots + \dot{T}^n}{n+1}$.

Since $\lim_{n \rightarrow \infty} \frac{\|\dot{T}^n\|}{n} = 0$,

$$(2.1) \quad (\dot{I} - \dot{T})\dot{P} = \lim_{n \rightarrow \infty} (\dot{I} - \dot{T}) \frac{\dot{I} + \dot{T} + \dots + \dot{T}^n}{n+1} = \lim_{n \rightarrow \infty} \frac{\dot{I} - \dot{T}^{n+1}}{n+1} = 0.$$

Thus $\dot{T}\dot{P} = \dot{P}$, and so

$$\dot{P}^2 = \lim_{n \rightarrow \infty} \frac{\dot{P} + \dot{T}\dot{P} + \dots + \dot{T}^n\dot{P}}{n+1} = \lim_{n \rightarrow \infty} \frac{(n+1)\dot{P}}{n+1} = \dot{P}.$$

Hence \dot{P} is an idempotent in $B(X)/K(X)$. By Lemma 2.6, there exists an idempotent P_0 in $B(X)$ such that $P - P_0 \in K(X)$. Replacing P with P_0 , we can assume without loss of generality that P is an idempotent in $B(X)$. Equation (2.1) also implies that $(I - T)P \in K(X)$, which means that $I - T$ is compact on PX . Hence $(I - T)^m$ is compact on PX .

We now show that PX is an essentially maximal subspace of X on which $(I - T)^m$ is compact. Suppose that $(I - T)^m$ is compact on a subspace M_0 of X containing PX . Let

$$f_n(z) := \frac{n + (n-1)z + (n-2)z^2 + \dots + z^{n-1}}{n+1}, \quad z \in \mathbb{C}, n \geq 1.$$

Note that $\dot{I} - \frac{\dot{I} + \dot{T} + \dots + \dot{T}^n}{n+1} = (\dot{I} - \dot{T})f_n(\dot{T})$. Therefore,

$$\dot{I} - \dot{P} = (\dot{I} - \dot{P})^m = \lim_{n \rightarrow \infty} f_n(\dot{T})^m (\dot{I} - \dot{T})^m,$$

and so

$$\lim_{n \rightarrow \infty} \|(I - P) - (f_n(T)^m (I - T)^m + K_n)\| = 0,$$

for some $K_1, K_2, \dots \in K(X)$.

Since $(I - T)^m$ is compact on M_0 , the operator $f_n(T)^m(I - T)^m$ is compact on M_0 and so is $f_n(T)^m(I - T)^m + K_n$ on M_0 . Thus $(I - P)|_{M_0}$ is the norm limit of a sequence in $K(M_0, X)$, and so $I - P$ is compact on M_0 . Since $PX \subset M_0$, we have that $(I - P)M_0 \subset M_0$. Therefore, $(I - P)|_{(I - P)M_0} = I|_{(I - P)M_0}$ is compact, and so $(I - P)M_0$ is finite dimensional. In other words, $\dim M_0/PX < \infty$.

“(2) \Rightarrow (1)”: By Proposition 2.5, $\widehat{(I - T)^m} = (I - \widehat{T})^m$ has closed range. Since by assumption $\lim_{n \rightarrow \infty} \frac{\|T^n\|_e}{n} = 0$, $\lim_{n \rightarrow \infty} \frac{\|\widehat{T}^n\|}{n} = \lim_{n \rightarrow \infty} \frac{\|\widehat{T}^n\|}{n} = 0$. By Mbekhta-Zemánek’s Theorem 1.2, the sequence $(M_n(\widehat{T}))_n$ converges in norm to an element in $B(\widehat{X})$. By Theorem 2.1, the result follows. \square

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